ECON 7010 - Macroeconomics I

Fall 2015

Notes for Lecture #7

Today:

- Firm Dynamics
- Non-stochastic Growth

Firm Dynamics

- capital: costs to adjusting capital stock (production disruptions, training, installation, etc)
- labor: costs to adjusting labor (e.g. training, search, etc)
- price: menu costs $\rightarrow$ costs to adjusting prices
- inventories (for finished and intermediate goods)
- organizational capital (knowledge and general operation of facility)

Capital Accumulation by a firm

- (note there is a choice here - we don't have to look at the firm - e.g., we could look at the country or a plant)
- Objective: maximize the discounted present value of profits
- $\beta < 1$ discount factor
  - using only β as a discount factor is a shortcut - really the discount factor depends on ore than just β (e.g., the ratio of marginal utilities)
- Profit flow per period: $\pi(A, k)$
  - k = capital stock
  - A = shock to profits (from demand or from technology)
- Source of $\pi(A, k)$:
  - $\pi(A, k) = \max_z R(A, k, z) - wz$
    - $k \equiv$ fixed factors of production (already paid for by investment in previous period $\Rightarrow$ no cost in intratemporal max problem)
    - $z \equiv$ flexible factors of production (e.g., labor, electricity)
    - $w \equiv$ input prices
    - $R(\cdot) \equiv$ Revenue function: $q(p(q))$ (q is quantity, p is price, p may depend on q if have lots of market share and so the market is not competitive)
    - $q(A, K, z) \equiv$ production function
  - Solving: $\max_z R(A, k, z) - wz \implies$ policy function $= z(A, k)$ (w here is a parameter, not a state variable)
  - This policy function them implies we can write: $R(A, k, z(A, k)) - wz(A, k) = \pi(A, k)$

Dynamic Programming Problem of the Firm

- $V(A,k) = \max_{k'} \pi(A,k) - p(\underbrace{k' - (1-\delta)k}_{\text{gross invest}}) - \underbrace{c(k',k)}_{\text{adj costs}} + \beta E_{A'|A} V(A',k'), \forall (A,k)$
  - Note the expectation is over the future profitability shocks, which are stochastic
- max k' because choosing investment today which implies capital tomorrow
- $p \equiv$ price of new capital
- $\delta \equiv$ depreciation rate on capital
- $I \equiv$ gross investment, $k' - k =$ net investment $= I - \delta k$
  - $\Rightarrow$ transition equation: $k' = k(1-\delta) + I$
  - intertemporal - how do I get productive capital tomorrow? By putting in investment today.
- On the price of capital (things to consider):
  - stochastic p
  - buying price > selling price
    - A bigger gap makes more cautious - not as responsive to increases in A
- Costs of adjustment
  - There are costs associated with changing the capital stock (i.e., making an investment)
  - We generally think of these costs as being one of two types:

  1. Convex costs of adjustment
     - cost more to increase capital a lot than a little
     - want to smooth, or spread out capital adjustment over time
     - "Rome not built in a day" - because expensive to adjust capital stock a lot
  2. Non-convex costs of adjustment
     - marginal cost of changing capital stock is not increasing
     - e.g. Fixed costs of adjustment
       - one-time cost to install as much capital as you want
       - $\Rightarrow$ bunching of activities (e.g. train all workers when shutdown plant to install new machines)
     - "Rome not built everyday"

  - Gap between buying and selling price come in here too - this gap may be modeled as a cost of adjustment
- <u>Quadratic Costs of Adjustment</u>
  - This is a special case of convex costs of adjustment (and the most frequently used)
  - $c(k',k) = \frac{\gamma}{2}\left(\frac{k'-(1-\delta)k}{k}\right)^2 k = \frac{\gamma}{2}\left(\frac{I}{k}\right)^2 \underbrace{k}_{\text{scaling with } k}$
  - Draw graph with c/k on one axis and I/k on other - and a parabola showing costs
  - FOC:
    - $\frac{\partial V}{\partial k'} : \underbrace{p + c_1(k',k)}_{\text{marginal cost}} = \underbrace{\beta E V_2(A',k')}_{\text{marginal benefit}}$
    - $\Rightarrow p + \gamma\left(\frac{I}{k}\right) = \beta E V_2(A',k') \Rightarrow \underbrace{\frac{I}{k}}_{\text{Invest Rate}} = \frac{1}{\gamma}[\beta E \underbrace{V_2(A',k')}_{\text{Marginal Q}} - p]$

- Marginal Q = the marginal value of the firm (how firm value changes for one more unit of capital)
- Marginal Q is unobservable
- With quadratic adjustment costs, the investment rate is proportional to the marginal value of the firm

* We can simplify the FOC by using the envelope condition:
  - $V_2(A,k) = \pi_2(A,k) + (1-\delta)p - c_2(k',k) + \frac{\partial k'}{\partial k}\left[\underbrace{-p - c_1(k',k) + \beta E_{A'|A} V_2(A',k')}_{=0,\text{ by the FOC}}\right]$
  - $\implies V_2(A,k) = \pi_2(A,k) + (1-\delta)p - c_2(k',k)$
  - $\implies V_2(A',k') = \pi_2(A',k') + (1-\delta)p - c_2(k'',k')$
  - Which means the FOC can be written as: $p + \gamma\left(\frac{I}{k}\right) = \beta[E_{A'|A}\pi_2(A',k') + (1-\delta)p - c_2(k'',k')]$

- Q-Theory
  - James Tobin (sometimes called "Tobins Q")
    * Q a sufficient statistic for firm invest
    * Q = marginal benefit of investment, so if adj cost+price capital < Q, invest.
    * invest up to point where Q=adj cost + price capital
    * measure using market value of firm and replacement cost of capital
  - Hayashi (Econometrica, 1982)
    * $\pi(A,k) = Ak$ (profit function proportional to capital stock) , and quadratic costs of adjustment, then marginal Q=Avg Q ($V_2(A,k) = \frac{V(A,k)}{k}$)
    * $\Rightarrow V(A,k) = \phi(A) * k$
    * $\Rightarrow \frac{I}{k} = \frac{1}{\gamma}[\text{Avg Q} - p]$
    * idea here is that can observe average q (look at stock market valuation firm capital stock)
    * Can then test q-theory because it implies investment rate a function of marginal q and an nothing else (e.g. proftiablilty)
    * q-theory an also be applied to determine firm investment or over/under valuation.
    * in practice, it's a little hard to measure because stock market volatility provides a lot of noise to get good estimates of the effect of avg Q

- More with quadratic costs of adj
  - Showing the Hayashi proof of Marginal Q = Avg Q under certain assumptions
  - Assume:
    * $\pi(A,k) = Ak$
    * Quadratic adjustment costs
    * Value function of the form $V(A,k) = \phi(A)k$ (this is a guess, but we'll prove it's true given the two assumptions above)
  - $V(A,k) = \max_I \underbrace{Ak}_{\text{profit fnc prop to k}} - pI - \underbrace{\frac{\gamma}{2}\left(\frac{I}{k}\right)^2 k}_{\text{quad costs of adj}} + \beta E_{A'|A} V(A',k')$
  - FOC: $\frac{\partial V}{\partial I} \Rightarrow \left(\frac{I}{k}\right) = \frac{1}{\gamma}\left[\beta E V_2(A',k') - p\right]$
  - Use the assumed value function to find V_2: $V_2(A,k) = \frac{\partial V(A,k)}{\partial k} = \phi(A)$
  - Which means we can write the FOC as: $\left(\frac{I}{k}\right) = \frac{1}{\gamma}[\beta E_{A'|A}\phi(A') - p] \equiv z(A)$
    * b/c $E_{A'|A}$, investment rate is only a function of A

    - $*$ How much investment we chose depends upon how well A informs us about A'
  - $\Rightarrow I = z(A)k$
  - Go through steps to get k' below...
  - $k' = (1-\delta)k + I$
  - $\implies k' = (1-\delta)k + z(A)k$
  - $\implies k' = ((1-\delta) + z(A))k$
  - Now, subbing in the FOC sol'n into the functional equation:
  - $\phi(A) * k = Ak - \underbrace{pz(A)k}_{I} - \underbrace{\frac{\gamma}{2}(z(A))^2 k}_{\text{cost of adj}} + \beta E_{A'|A}\{\phi(A')((1-\delta) + z(A))k\}$
  - $[\phi(A)]k = k[A - pz(A) - \frac{\gamma}{2}(z(A))^2 + \beta E_{A'|A}\{\phi(A')(1 - \delta + z(A))\}]$
    - $*$ holds for all k b/c k multiplies both sides (i.e., k cancels out)
    - $*$ we know this is a sol'n to the problem
    - $*$ $V(A, k)$ for firm should equal the stock market value

DRAW "Economy's problem" - household and firms interact through market, produces a decentralized allocation.

- The planner's problem (a dynamic optimization problem) is related to the market allocation by the Fundamental Welfare Theorems
  1. Markets give Pareto efficient outcomes.
  2. Can get any Pareto outcome with lump sum transfer and let market work.
- We will use this relationship between the planner's problem and the decentralized solution
- In particular, we will use solve the planner's problem of the growth model instead of the decentralized problem
  - This means we do not have to solve for the market eq'm all along the growth path
  - It's much simpler!
  - And by the welfare theorems, we know that this is a potential decentralized outcome (at least if assumptions hold)
  - And even if assumptions do not hold, we can use this sol'n to compare to the decentralized solution (i.e., compare market outcome to 1st best outcome)

<u>Growth Model</u> (a popular general equilibrium model)

- May be stochastic or not (we'll focus on the non-stochastic)
- $k \equiv$ stock of capital (per capita, if you wish)
- transition equation: $k' = k(1-\delta) + i$
  - δ is the rate of physical depreciation of the capital stock
- Resource constraint: $\underbrace{y}_{\text{real output}} = \underbrace{c}_{\text{real consumption}} + i$ {This is a 1 sector model}
- Production function: $y = f(k)$

- Objective function: $\max_{c_t} \sum_{t=0}^{\infty} \beta^t u(c_t)$
- Assumptions:
  - $u'(\cdot) > 0, u''(\cdot) < 0, u'(0) = \infty$
  - $f'(\cdot) > 0, f''(\cdot) < 0, f'(0) = \infty, f'(\infty) = 0, f(0) = 0$
  - $0 < \beta < 1$ (Discounting)
  - Need these assumptions for a solution to exist
- We are abstracting from labor supply/demand decisions (i.e., we solve the planner's problem, not the decentralized sol'n)
- To simplify, we assume no disutility of labor
- We care only about "real things" - investment, capital, consumption, output

Planner's Dynamic Programming Problem

- $V(k) = \max_{0 \leq k' \leq k(1-\delta)+f(k)} u(c) + \beta V(k'), \forall k \in [0, \bar{k}]$
  - state: k
  - control: k'
  - transition equation: $k' = k(1-\delta) + i$, δ = rate of physical depreciation
  - Resource contraint: $i = y - c$
  - Technology: $y = f(k)$
  - $\Rightarrow k' = (1-\delta)k + f(k) - c$
- How do we know there exists a $\bar{k}$ to bound the problem?
  - Consider the difference equation that describes the evolutions of k:
    * $k_{t+1} = (1-\delta)k_t + \underbrace{f(k_t) - c(k_t)}_{i_t}$
  - Can use this equation to find $\bar{k}$ by considering the case where $c_t = 0$ (i.e., spend all resources on investment)
  - In this case, let $H(k_t) = k_{t+1} = (1-\delta)k_t + f(k_t)$
  - Solve this for $\bar{k}$: $\bar{k} = (1-\delta)\bar{k} + f(\bar{k})$
  - Graphically: Draw $H(k)$ and k axes and show that $H(k)$ function crosses 45 degree line (say that we assume this picture so we can "put it in a box"), k needs to be bounded so we need $f'(k) > 0$ and $f''(k) < 0$ and that still doesn't guarantee $H(k)$ crosses the 45 degree line. Note that $c = 0$ in this picture since all output going to gross investment.
  - This is how we bound the problem - even if consumer zero, we are still limited to $\bar{k}$
  - the assumptions we need for this boundedness (i.e., $H(k_t)$ crossing the 45 degree line) are:
    * $f'(k) > 0, f''(k) < 0, H(0) = 0, f'(0) = \infty$
    * $\lim_{k \to \infty} f'(k) = 0$ - this one condition is sufficient for $H(k_t)$ to cross the 45 degree line
    * note that $H'(k) = f'(k) + (1-\delta)$ so as $f'(k)$ goes toward zero, know slope of $H(k)$ will fall below 1
- can rewrite as: $V(k) = \max_{k'} u(k(1-\delta) + f(k) - k') + \beta V(k')$
- Policy Function:

- $k' = \phi(k)$
- $c = k(1-\delta) + f(k) - \phi(k) \equiv \Psi(k)$
- for an interior solution, the Euler equation is:
  * $\frac{\partial V}{\partial k'} \Rightarrow u'(c) = \beta V_{k'}(k')$ or
  * $\underbrace{u'(k(1-\delta)+f(k)-\phi(k)) = \beta V_{k'}(\phi(k))}_{\Rightarrow \phi(k) \text{is increasing and has slope <1 at steady state}}, \forall k \in [0, \bar{k}]$
- Proposition: $\phi(k)$ is an increasing function.
- Proof: If $V(k)$ is strictly concave (and it's not free to assume this - if follows from our assumption that $u'' < 0$ (Stokey and Lucas prove this)) then that $\phi(k)$ is increasing is direct from the Euler above.
  * just look at the above, b/c $u'' < 0$, $f'(k) > 0$, then when $k \uparrow$,
  * $u'(\underbrace{k(1-\delta)}_{\uparrow} + \underbrace{f(k)}_{\uparrow} - \phi(k)) = \beta V_{k'}(\phi(k))$
  * so if $\phi(k) \downarrow$, then $c \uparrow$, then $u'(c) \downarrow$, and $\beta V_{k'}(\phi(k)) \uparrow$, which can't happen
  * This implies that $\phi(k) \uparrow$ when $k \uparrow$
- Graphically: Draw graph with k on horizontal and $\phi(k) = k'$ on the vertical. Have 45 degree line. Show $\phi(k)$ function crossing 45 degree line at k^*. Point out that $\phi(k)$ is increasing with slope less than one.

Steady State

- The steady state is the eq'm of the growth model
- In the SS, all variables are constant (or grow at same rate)
- k^* denotes the steady state level of capital
- $k^* = \phi(k^*) \rightarrow$ like a fixed point theorem
- What is k^*? - go to Euler to get the answer
  - $u'(c) = \beta V'(k')$
  - $\frac{\partial V(k)}{\partial k} \Rightarrow V'(k) = u'(c)[f'(k) + (1-\delta)] \rightarrow$ by the envelope condition
  - $\Rightarrow V'(k') = u'(c')[f'(k') + (1-\delta)]$
  - $\Rightarrow u'(c) = \beta V'(k'))$
  - $\Rightarrow u'(c) = \beta u'(c')[f'(k') + (1-\delta)]$
    * How to think of RHS of Euler:
    * Recall, in HH problem: $u'(c) = \beta R u'(c')$
    * R is analogous to $(f'(k') + (1-\delta))$ in the planner's problem (if don't consume in the household, you get a return to savings, R. If you don't consume in the planner's problem, get added capacity/output later.
  - Steady state (SS): $c_t = c^*$, $k_t = k^*$, $\forall t$ (consume the same, always)
  - Euler: $u'(c) = \beta u'(c')[f'(k') + (1-\delta)]$
  - In SS: $u'(c^*) = \beta u'(c^*)[f'(k^*) + (1-\delta)]$
  - $\Rightarrow 1 = \beta[f'(k^*) + (1-\delta)]$
  - Equation gives steady state for capital stock (solve it to find k^*) $\implies \frac{1}{\beta} - 1 + \delta = f'(k^*)$
  - This happens because we have $c_t = c^*$, $\forall t$, then cancel out $u'(c)$ and $u'(c')$.

- $\Rightarrow c^* = f(k^*) - \delta k^*$
- DRAW: graph with k on horizontal and $\phi(k) = k'$ on the vertical axis. Have 45 degree line. Show how k goes to SS from any initial k_1

<u>Solving the non-stochastic growth model</u>

- Two ways:
  1. Special cases (i.e., certain functional forms where can solve by hand)
     - $u(c) = ln(c), \delta = 1, f(k) = k^\alpha (0 < \alpha < 1), 0 < \beta < 1$
     - $\Rightarrow V(k) = A + Bln(k), B = \frac{\alpha}{1-\beta\alpha}$ (work it out - just like cake eating problem)
  2. Value function iteration (most cases)
     - $V_{i+1}(k) = \max_{k'} u(c) + \beta V_i(k')$
     - $c = k(1-\delta) + f(k) - k'$
     - Grow.m matlab program
- Solution to planner's problem:
  - $k' = \phi(k)$: policy function
    - Maps from current state into the control
    - Increasing in k (as we showed earlier)
  - $V(k)$: value function

<u>Properties of $V(k)$:</u>

1. Exists, unique, found by value function iteration
2. Increasing
3. Strictly concave (proof in Adda-Cooper book, S-L book - gist of it is that b/c $u'' < 0$)
   - FOC: $u'(c) = \beta V'(k')$
   - $V(k)$ strictly concave $\Rightarrow \frac{\partial k'}{\partial k} = \phi'(k) > 0$

<u>Properties of the SS, k^*.</u> (nontrivial ones $\rightarrow c^*, k^* > 0$)

- There is a unique, nontrivial steady-state
- $k^* < \bar{k}$
  - Proof: $\bar{k}$ (max k is where spend all resources on investment) solves $\frac{f(\bar{k})+(1-\delta)\bar{k}}{\bar{k}} = \frac{\bar{k}}{\bar{k}} \Rightarrow \frac{f(\bar{k})}{\bar{k}} + (1 - \delta) = 1$
  - $\Rightarrow \beta[f'(\bar{k}) + (1-\delta)] < 1$ b/c $\beta < 1$ and b/c $\frac{f(\bar{k})}{\bar{k}} > f'(\bar{k})$ (b/c since concave, avg product greater than marginal product at any particular point)
  - So $\beta[f'(\bar{k}) + (1-\delta)] < 1 = \beta[f'(k^*) + (1-\delta)]$ (from Euler above)
  - $\Rightarrow f'(\bar{k}) < f'(k^*)$
  - $\Rightarrow \bar{k} > k^*$ b/c $f(\cdot)$ concave

- Draw graph with k' and k as axes. Show 45 degrees line and S-curve of $\phi(k)$ policy function. Say that $\phi(\bar{k}) < \bar{k}$ (b/c if at $\bar{k}$, that means $c = 0$, which is not optimal. Show how converge to k^* from above.
- Draw another graph with k' and k as axes. Show 45 degrees line and S-curve of $\phi(k)$ policy function. Show how converge to $k*$ from below. Show how growth faster for poor economies.
- What aspects of the economy determine how fast you get to k^*? i.e., what is the cost of getting to k^*?
  - Consumption smoothing
    - The more curvature there is in $u(\cdot)$, the more costly it is to get to $k^* \Rightarrow$ a slower transition to k^*
    - $u(\cdot)$ doesn't affect k^*, but it does affect getting there. (recall that k^* solves: $1 = \beta[f'(k)+1-\delta]$ – there is no $u(\cdot)$ here...)
    - Draw two graphs with k and k' axes. On with lots of curvature in $\phi(k)$ and one with little. Show how convergence slower with more curvature. Point out that $A(c) = \frac{-u''(c)}{u'(c)}$ (coefficient of absolute risk aversion is lower in one with little curvature).
  - Production function, $f(k)$
    - It is concave by assumption
    - More curvature means lower k^*, slower growth
  - Depreciation. δ
    - High depreciation rates $\Rightarrow$ lower k^*
    - High depreciation rates $\Rightarrow$ slower growth
  - Rate of time preference, β
    - Less patience $\Rightarrow$ lower k^*
    - Less patience $\Rightarrow$ slower growth

<u>Leading example for non stochastic growth:</u>

- $u(c) = ln(c)$
- $f(k) = k^\alpha$
- $\delta = 1$
- $\Rightarrow$ transition equation: $c = f(k) + (1-\delta)k - k' = f(k) - k' = k^\alpha - k'$
- (As with the cake eating problem on this form) Guess: $V(k) = A + Bln(k)$. Now try to prove this:
  - In general, $V(k) = u(f(k) + (1-\delta)k - k') + \beta V(k')$
  - With assumptions and guess, this becomes: $A + Bln(k) = ln(k^\alpha - k') + \beta[A + Bln(k')]$
  - The FOC is thus: $\frac{1}{k^\alpha - k'} = \beta B \frac{1}{k'}$
  - This then implies that $k' = \beta K^\alpha \left(\frac{B}{1+\beta B}\right)$
  - Plugging this policy function back into the FE, we get: $A + Bln(k) = ln\left(k^\alpha - \beta k^\alpha \left(\frac{B}{1+\beta B}\right)\right) + \beta\left[A + Bln\left(\beta k^\alpha\left(\frac{B}{1+\beta B}\right)\right)\right]$
  - with some algebra, we can then get to: $A + Bln(k) = \alpha(1+\beta B)ln(k) + \beta A + \beta Bln(\beta B) - \beta Bln(1+\beta B)$
  - This implies that $B = \frac{\alpha}{1-\beta\alpha}$, $A = \frac{\beta B(ln(\beta B) - ln(1+\beta B)}{1-\beta}$, so the conjecture does indeed work out.